Sufficient conditions for the Sal'nikov equation to have at least two limit cycles

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We show that the Sal'nikov thermokinetic oscillator system has at least two limit cycles, an unstable one lying inside a stable one. The proof uses the elementary technique of Liénard's equation.

1. Introduction

In this paper we deal with the Sal'nikov thermokinetic oscillator system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu - Kx\exp\left(\frac{y}{1+\xi y}\right),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + x\exp\left(\frac{y}{1+\xi y}\right), \tag{1.1}$$

and give some sufficient conditions on μ, K, ξ under which the system (1.1) has at least two limit cycles. This was first proved by Gray and Roberts [2] and Kay and Scott [4], then by Forbes [1]. All proofs involve the use of Hopf bifurcation and numerical methods, and the stability of the limit cycles is determined by means of Floquet theory.

Here we want to give a new proof. For this we transform (1.1) into a Liénard equation, and we use a result obtained by Zhang Zhifen [5].

2. Two limit cycles

Clearly, the system (1.1) has only a unique equilibrium point $M(\bar{x}, \bar{y})$, where

$$\bar{x} = \bar{y}\exp\left(-\frac{\bar{y}}{1+\xi\bar{y}}\right),$$
$$\bar{y} = \frac{\mu}{K}.$$

Let K_1, K_2 be constant and $K_1 > \bar{y}, K_2 > \bar{y} + K_1/K$.

Define

$$\Delta = \left\{(x,y)|0 \leqslant x \leqslant K_1, 0 \leqslant \frac{1}{K}x + y \leqslant K_2\right\}.$$

It is not difficult to prove the following.

LEMMA 1

The area Δ is a positive invariant set for system (1.1) in $\mathbb{R}^2$.

In order to prove that system (1.1) has at least two limit cycles, let us consider:

PROPOSITION 1

The planar differential system (1.1) can be transformed into the following equation of Liénard type:

$$\frac{dx}{d\tau} = y - F(x),$$
$$\frac{dy}{d\tau} = -q(x), \tag{1.2}$$

by a change of variables.

Proof

First, we translate the equilibrium point $M(\bar{x}, \bar{y})$ to the origin by setting

$$X = x - \bar{x}, \quad Y = y - \bar{y}.$$

System (1.1) transforms into

$$\frac{dX}{dt} = \mu - K(X+\bar{x})\exp\left[\frac{Y+\bar{y}}{1+\xi(Y+\bar{y})}\right],$$
$$\frac{dY}{dt} = -(Y+\bar{y}) + (X+\bar{x})\exp\left[\frac{Y+\bar{y}}{1+\xi(Y+\bar{y})}\right]. \tag{1.3}$$

Now let

$$Z = -(Y+\bar{y}) + (X+\bar{x})\exp\left[\frac{Y+\bar{y}}{1+\xi(Y+\bar{y})}\right].$$

Then (1.3) becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = Z,$$
$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\Psi_0(Y) - \Psi_1(Y)Z - \Psi_2(Y)Z^2, \tag{1.4}$$

where

$$\Psi_0(Y) = KY \exp\left[\frac{Y+\bar{y}}{1+\xi(Y+\bar{y})}\right],$$
$$\Psi_1(Y) = 1 - \frac{Y+\bar{y}}{[1+\xi(Y+\bar{y})]^2} + K \exp\left[\frac{Y+\bar{y}}{1+\xi(Y+\bar{y})}\right],$$
$$\Psi_2(Y) = -\frac{1}{[1+\xi(Y+\bar{y})]^2}.$$

Introducing the new time transformation

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \phi(Y) = C_0 \exp\left[\frac{-(Y+\bar{y})}{1+\xi(Y+\bar{y})}\right],$$

where

$$C_0 = \exp\left(\frac{\bar{y}}{1+\xi\bar{y}}\right),$$

we obtain the Liénard equation

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\tau^2} + f(Y)\frac{\mathrm{d}Y}{\mathrm{d}\tau} + q(Y) = 0, \tag{1.5}$$

where

$$f(Y) = \Psi_1(Y)\phi(Y),$$
$$q(Y) = \Psi_2(Y)\phi^2(Y), \tag{1.6}$$

or rather the equivalent system (1.2), with

$$F(x) = \int_0^x f(\tau)\,\mathrm{d}\tau.$$

PROPOSITION 2

[1] The Sal'nikov thermokinetic oscillator problem defined in equations (1.1) has no periodic solutions when $\xi > \frac{1}{4}$, for all μ and K.

Proof

The divergence of the vector field defined by (1.2)

$$\nabla = \frac{\partial}{\partial x}[y - F(x)] + \frac{\partial}{\partial y}[-q(x)] ,$$
$$= -\left\{\frac{\xi^2(x+\bar{y})^2 + (2\xi - 1)(x+\bar{y}) + 1}{[1+\xi(x+\bar{x})]^2}\right\}\phi(x) ,$$

is negative, since the discriminant of

$$p(u) = \xi^2u^2 + (2\xi - 1)u + 1$$

is negative for $\xi > \frac{1}{4}$ and $\phi(x) > 0$.

Hence by Bendixson's negative criterion [3] there is no periodic orbit in (1.2) or equivalently (1.1).

Now, as $\xi \leqslant \frac{1}{4}$, there exists two roots for the equation

$$\xi^2(x+\bar{y})^2(2\xi - 1)(x+\bar{y}) + 1 = 0 .$$

Let x_1, x_2 represent the two roots; then we have

$$x_1 = \hat{x}_1 - \bar{y}, \quad x_2 = \hat{x}_2 - \bar{y} ,$$

where $0 < \hat{x}_1 < \hat{x}_2$, and

$$\hat{x}_i = \frac{1 - 2\xi \mp \sqrt{1 - 4\xi}}{2\xi^2}, \quad i = 1, 2 .$$

Define

$$\Omega = \{\xi | 0 < \xi \leqslant \tfrac{1}{4} \quad \text{and} \quad \hat{x}_1 < \bar{y} < \hat{x}_2\} ,$$

$$\Pi(x) = \frac{\xi^2(x - x_1)(x - x_2)}{[1+\xi(x+\bar{y})]^2}\exp\left[\frac{-(x+\bar{y})}{1+\xi(x+\bar{y})}\right] .$$

Obviously, $\Pi(x) > 0$ as $x \in (-\infty, x_1) \cup (x_2, +\infty)$, and $\Pi(x) < 0$ as $x \in (x_1, x_2)$.

By (1.2), (1.4) and (1.6), we have

$$F(x) = \int_0^x f(x)\, dx = \int_0^x C_0[\Pi(x) + K]\, dx . \tag{1.7}$$

LEMMA 2

[5] Consider the system (1.2), and let hypotheses

A1. $q, f \in C^0(|x| < +\infty)$,

A2. $xq(x) > 0, x \neq 0$,

hold. Then the positive half trajectory and negative half trajectory passing the point $A(x, F(x))(x \neq 0)$ must intersect the y-axis or tend toward the origin for system (1.2).

LEMMA 3

[5] Consider the system (1.2), and let hypotheses

B1. $f, q \in C^0(|x| < +\infty)$,
B2. $xq(x) > 0, x \neq 0$,
B3. either

$$\overline{\lim_{x \to +\infty}} F(x) = +\infty,$$

$$\underline{\lim}_{x \to -\infty} F(x) = -\infty,$$

or

$$\int_0^{\pm} q(x)\,\mathrm{d}x = Q(\pm\infty) = +\infty,$$

$$F(x) > K_1, \quad x > 0,$$

$$F(x) < K_2, \quad x < 0,$$

hold. Then the positive half trajectory passing the point $P(0, y_p)(y_p \neq 0)$ must intersect the curve $y = F(x)$ for system (1.2).

We obtain the following result.

THEOREM 1

Consider the system of equations (1.2). If $\xi \in \Omega, K \geqslant 1$, and there exists a constant $M > \max\{|x_1|, x_2\}, e_2 < e_1$, such that $F(x) > e_1$ as $x > M$ and $F(x) < e_2$ as $x < -M$, then system (1.2) has at least one stable limit cycle and one unstable limit cycle in the area Δ.

Proof

By (1.7)

$$f(x) = C_0[\pi(x) + K].$$

Choose $K \geqslant 1$ (i.e. $1/K \leqslant 1$), then we have

$$f(x) > 0 \tag{1.8}$$

as $x \in (x_1, x_2)$.

Define the energy function

$$\lambda(x, y) = \int_0^x q(x)\,\mathrm{d}x + \tfrac{1}{2}y^2.$$

Then, for system (1.2), we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = q(x)\dot{x} + y\dot{y} = -q(x)F(x)$$
$$= -KC_0 x \exp\left[-\frac{x + \bar{y}}{1 + \xi(x + \bar{y})}\right] \int_0^x f(x)\,\mathrm{d}x < 0 \tag{1.9}$$

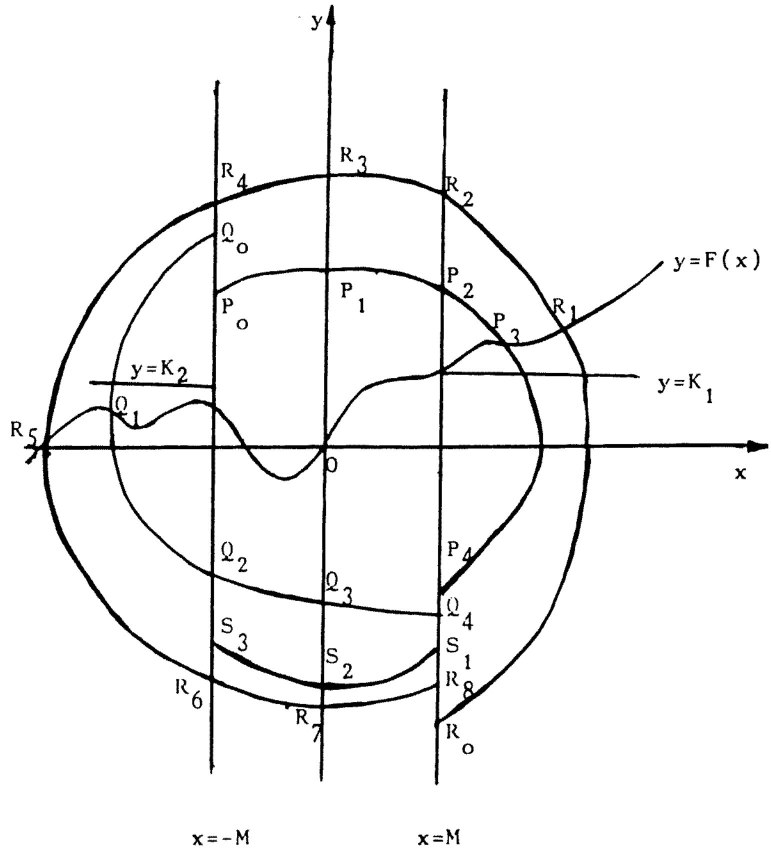

Fig. 1. Illustrating the proof of theorem 1.

as $x\in(x_1,x_2)$. Hence

$$\frac{d\lambda}{d\tau}<0, \quad 0<|x|\leqslant 1. \tag{1.10}$$

So, the equilibrium point $\bar{O}(0,0)$ is stable, and the closed curve $L_1:\lambda(x,y) = C(0<C\leqslant 1)$ is regarded as an interior boundary curve for system (1.2).

Now we will construct an exterior boundary curve for system (1.2).

Define

$$e=\max_{|x|\leqslant M}|F(X)|, \quad b=\max_{|x|\leqslant M}|q(x)|.$$

As $d\geqslant 1$ (i.e., $1/d\leqslant 1$), we have for system (1.2)

$$\frac{dx}{d\tau}=y-F(x)>a=d-e>0,$$

as $|x| \leqslant M, y > d$, and

$$\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| = \left|\frac{-q(x)}{y - F(x)}\right| \leqslant \frac{b}{a} \leqslant 1\,, \tag{1.11}$$

$(|x| \leqslant M)$.

Choose a point $P_0(x_{P_0}, y_{P_0})$ on the line $x = -M$ such that $y_{P_0} > 2d$. Then $b/a \leqslant 1$ as $d \geqslant 1$. Thus the positive half trajectory $\Gamma^+(P_0)$ passing the point P_0 must intersect the y-axis at P_1 and intersect the line $x = M$ at $P_2(x_{P_2}, y_{P_2})$. We have along the path $\widehat{P_0P_1P_2}$:

$$y_{\widehat{P_0P_1P_2}} > d, \quad |y_{P_2} - y_{P_0}| \leqslant 1\,. \tag{1.12}$$

By lemma 2, $\Gamma^+(P_0)$ will intersect the curve $y = F(x)$ at P_3. By lemma 3, $\Gamma^+(P_0)$ will intersect the line $x = M$ at $P_4(x_{P_4}, y_{P_4})$ again.

Now we choose a point $S_1(x_{S_1}, y_{S_1})$ on the line $x = M$ such that $y_{S_1} < \min(-2d, y_{P_4})$. Similarly, the positive half trajectory $\Gamma^+(S_1)$ passing the point S_1 will intersect the y-axis at S_2 and intersect the line $x = -M$ at S_3. Along the curve $\widehat{S_1S_2S_3}$, we have

$$y_{\widehat{S_1S_2S_3}} < -d, |y_{S_3} - y_{S_1}| \leqslant 1\,, \tag{1.13}$$

as $d \geqslant 1$.

On the line $x = -M$, we choose a point $Q_0(x_{Q_0}, y_{Q_0})$ such that $Y_{Q_0} > Y_{P_0}$. Similarly, the negative half trajectory $\Gamma^-(Q_0)$ passing the point Q_0 will intersect the curve $y = F(x)$ at Q_1 and intersect the line $x = -M$ at Q_2, the negative y-axis at Q_3, and intersect the line $x = M$ at $Q_4(x_{Q_4}, y_{Q_4})$.

If $y_{Q_4} \geqslant y_{P_4}$, then we define the closed curve

$$\bar{L}_2 = \widehat{P_4P_3P_0} \cup \overline{P_0Q_0} \cup \widehat{Q_0Q_1Q_4} \cup \overline{Q_4P_4}\,,$$

and the exterior boundary curve L_2 for system (1.2) is made by L_2. So a ring domain Ω is made by the curve L_1 and L_2 for system (1.2). By the Poincaré–Bendixsons theorem [3], system (1.2) or (1.1) has at least an unstable limit $\bar{\Gamma}$ in $\bar{\Omega}$. By lemma 1, we have $\bar{\Gamma} \subset \Delta$.

We note that, for system (1.2), we have on the line $\overline{P_0Q_0}$:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -q(-M) > 0\,, \tag{1.14}$$

and on the line $\overline{Q_4P_4}$:

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -q(M) < 0\,. \tag{1.15}$$

If $y_{Q_4} < y_{P_4}$, then we choose a point $R_0(x_{R_0}, y_{R_0})$ on the line $x = M$ such that

$$y_{R_0} < \min\{Q_4, S_1, -3d\}\,.$$

The negative half trajectory $\Gamma^-(R_0)$ passing the point R_0 will intersect the curve

$y = F(x)$ at R_1, and also the line $x = M$ at R_2, and the positive y-axis at R_3, and also the line $x = -M$ at $R_4(x_{R_4}, y_{R_4})$, as $d \geqslant 1$.

If $y_{R_4} \leqslant y_{Q_0}$, then define the closed curve

$$\overline{\overline{L}}_2 = \widehat{R_0R_1R_4} \cup \overline{R_4Q_0} \cup \widehat{Q_0Q_1Q_4} \cup \overline{Q_4R_0}\,.$$

Paying attention to (1.14) and (1.15), we know that the ring domain $\overline{\overline{\Omega}}$ is made by two closed curves L_1 and $\overline{\overline{L}}_2$ for system (1.2). By the Poincaré–Bendixsons theorem [3], system (1.2) or (1.1) has at least an unstable limit cycle $\overline{\overline{\Gamma}}$ in $\overline{\overline{\Omega}}$, and $\overline{\overline{\Gamma}} \subset \Delta$ by lemma 1.

If $y_{R_4} > y_{Q_0}$, then $\Gamma^-(R_0)$ will intersect the curve $y = F(x)$ at R_5, and also the line $x = -M$ at R_6, the negative y-axis at R_7 and the line $x = M$ at R_8.

If $y_{R_8} \geqslant y_{S_1}$, then $y_{R_8} > y_{R_0}$. Define

$$\overline{\overline{\overline{L}}}_2 = \widehat{R_0R_3R_7R_8} \cup \overline{R_8R_0}\,.$$

Similarly, the ring domain is made by L_1 and $\overline{\overline{\overline{L}}}_2$ for system (1.2).

If $Y_{R_8} < Y_{S_1}$, then we define the energy function

$$\bar{\lambda}(x, y) = \tfrac{1}{2}(y - K_1)^2 + \int_0^x q(x)\,\mathrm{d}x\,.$$

Along the path $\widehat{R_0R_2R_4R_6R_8}$ we have

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}y} = \frac{q(x)\dot{x} + (y - K_1)\dot{y}}{\dot{y}}\,,$$
$$= F(x) - K_1\,.$$

So,

$$\bar{\lambda}_{R_6} - \bar{\lambda}_{R_4} = \int_{y_{R_4}}^{y_{R_6}} [F(x) - K_1]\,\mathrm{d}y = \int_{y_{R_6}}^{y_{R_4}} [K_1 - F(x)]\,\mathrm{d}y$$
$$> (K_1 - K_2)(y_{R_4} - y_{R_6}) > 0\,; \tag{1.16}$$

$$\bar{\lambda}_{R_2} - \bar{\lambda}_{R_0} = \int_{y_{R_0}}^{y_{R_2}} [F(x) - K_1]\,\mathrm{d}x > 0\,;$$
$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}x} = \frac{q(x)[K_1 - F(x)]}{y - F(x)}\,. \tag{1.17}$$

We know that as $d \geqslant 1$, we have

$$\left|\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}x}\right| \leqslant 1, \quad |x| \leqslant M\,. \tag{1.18}$$

So, as $d \geqslant 1$, $|x| \leqslant M$, we obtain

$$|\bar{\lambda}_{R_8} - \bar{\lambda}_{R_6}| = \left| \int_{X_{R_6}}^{X_{R_8}} \frac{q(x)[K_1 - F(x)]}{y - F(x)} \mathrm{d}x \right| \leqslant 1 ; \tag{1.19}$$

$$|\bar{\lambda}_{R_4} - \bar{\lambda}_{R_2}| = \left| \int_{X_{R_2}}^{X_{R_4}} \frac{q(x)[K_1 - F(x)]}{y - F(x)} \mathrm{d}x \right| \leqslant 1 . \tag{1.20}$$

By (1.16)–(1.20), we obtain

$$\begin{aligned} \bar{\lambda}_{R_8} - \bar{\lambda}_{R_0} &= (\bar{\lambda}_{R_8} - \bar{\lambda}_{R_6}) + (\bar{\lambda}_{R_6} - \bar{\lambda}_{R_4}) + (\bar{\lambda}_{R_4} - \bar{\lambda}_{R_2}) + (\bar{\lambda}_{R_2} - \bar{\lambda}_{R_0}) \\ &> (K_1 - K_2)(y_{R_4} - y_{R_6}) - |\bar{\lambda}_{R_8} - \bar{\lambda}_{R_6}| - |\bar{\lambda}_{R_4} - \bar{\lambda}_{R_2}| \\ &> \tfrac{1}{2}(K_1 - K_2)(y_{R_4} - y_{R_6}) > 0 . \end{aligned} \tag{1.21}$$

By (1.21), we have

$$y_{R_8} > y_{R_0} \tag{1.22}$$

as $d \geqslant 1$.

Define

$$L_2 = \widehat{R_0 R_2 R_4 R_6 R_8} \cup \overline{R_8 R_0} .$$

By considering (1.15), we know that the ring domain Ω_1 is made by the curve L_1 and L_2. Using the Poincaré–Bendixson theorem [3], system (1.2) or (1.1) has at least an unstable limit cycle Γ_1 in Ω_1. Lemma 1 shows that the limit cycle $\Gamma_1 \subset \Delta$.

Lastly, the Γ_1 is regarded as a interior boundary curve and the closed curve $\partial(\Delta)$ (i.e.: the boundary of the area Δ) is regarded as a exterior boundary curve. Then the ring domain Ω_2 is made by Γ_1 and $\partial(\Delta)$.

By the Poincaré–Bendixson theorem again, system (1.2) or (1.1) has at least a stable limit cycle Γ_2 in the area Ω_2.

Therefore, system (1.2) or (1.1) has at least two limit cycles in the area Δ and the proof is completed.

Remark 1

Via Hopf bifurcation theory, there are regions of parameter space where the Sal'nikov thermokinetic oscillator (1.1) exhibits one stable limit cycle [2]. In another point of view, the uniqueness of limit cycles of the system (1.1) follows from an application of the Zhang Zhifen theorem [6] to Liénard's system (1.2).

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